Quantum Helicity Entropy of Moving Bodies

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Lorentz transformation of the reduced helicity density matrix for a massive spin $\frac{1}{2}$ particle is investigated in the framework of relativistic quantum information theory for the first time. The corresponding helicity entropy is calculated, which shows no invariant meaning as that of spin. The variation of the helicity entropy with the relative speed of motion of inertial observers, however, differs significantly from that of spin due to their distinct transformation behaviors under the action of Lorentz group. This novel and odd behavior unique to the helicity may be readily detected by high energy physics experiments. The underlying physical explanations are also discussed.

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Quantum information theory is usually formulated in the framework of non-relativistic quantum mechanics, since particles moving at relativistic speeds may not be needed to realize the promise of quantum information process such as quantum computation. However, relativity, especially special relativity plays a significant role in quantum entanglement and related quantum technology, such as teleportation. This point is obviously justified by quantum optics, which is well established on the basis of not only quantum theory but also special relativity in nature[1]. For example, most of EPR-type experiments have been performed by photon pairs[2, 3]. In addition, experiments of quantum teleportation have also been extensively carried out by photons[4, 5].

Recently, in particular, considerable efforts have been expanded on the theoretical investigation of quantum information theory in relativistic framework, which has gone beyond from photons to electrons, and from explicit examples calculated in some specific cases to general framework formulated in relativistic quantum mechanics and even relativistic quantum field theory[6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. For review, please refer to [18]. A central topic in this interesting and active research field is whether quantum entanglement is observer-dependent. Especially, for a pure one particle state, it has been shown that the reduced spin density matrix remains no covariant between inertial observers with relative motion, and the corresponding spin entropy is not an invariant scalar except in the limiting case of sharp momenta[8].

However, as inferred above, for Dirac fields, previous discussions of relativistic quantum information theory focus primarily on quantum entanglement between spin and momentum degrees of freedom. On the other hand, since the helicity has an advantage in providing a smooth transition to the massless case, it is the helicity rather than spin that is more often under both theoretical consideration and experimental detection in high energy physics. Although both the helicity states and the spin states can constitute the basis of Hilbert space of one particle, they differ in the way of unitary transformation under the action of Lorentz group[37]. As a result, the entanglement properties for helicity differs remarkably from those for spin after we trace out the momentum degree of freedom. Especially, it is found that in the sharp momentum limit, unlike the vanishing spin entropy, at small velocities of the inertial observer the helicity entropy demonstrates a sudden jump onto a constant value, half of the entropy for the maximal entangled Bell states, which may be easily detected in high energy physics experiments. Thus for both theoretical completeness and possible implementation in high energy physics, it is intriguing and significant to investigate quantum entanglement between helicity and momentum in relativistic framework. In this paper, we shall make a first step toward investigation of this important but ignored issue.

Start with a field with positive mass m and spin $\frac{1}{2}$, we can also construct the helicity states $|p; \lambda\rangle$ as a complete orthonormal basis for Hilbert space of one particle. The unitary operator $U(\Lambda)$ acting on these helicity states for

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a Lorentz transformation Λ gives[37]

$$\begin{aligned}
&U(\Lambda)|p;\lambda\rangle \\
&= \sqrt{\frac{(\Lambda p)^0}{p^0}} D_{\lambda'\lambda}[R^{-1}(\Lambda p)L^{-1}(\Lambda p)\Lambda L(p)R(p)]|\Lambda p;\lambda'\rangle \\
&= \sqrt{\frac{(\Lambda p)^0}{p^0}} D_{\lambda'\lambda}[B^{-1}(\Lambda p)R^{-1}(\Lambda p)\Lambda R(p)B(p)]|\Lambda p;\lambda'\rangle \\
&= \sqrt{\frac{(\Lambda p)^0}{p^0}} D_{\lambda'\lambda}[Z(\Lambda, p)]|\Lambda p;\lambda'\rangle. \qquad (1)
\end{aligned}$$

Here $R(p)$ is the rotation that carries the z axis into the direction $\mathbf{p}$, $B(p)$ is the boost from rest to the momentum $|\mathbf{p}|$ in the z direction, and $L(p)$ is the pure boost from rest to the momentum $\mathbf{p}$. Obviously, $L^{-1}(\Lambda p)\Lambda L(p)$ is just Wigner rotation, usually denoted by $W(\Lambda, p)$. In addition, D is the spin $\frac{1}{2}$ irreducible unitary representation of Lorentz group. Note that these helicity states differ in way of unitary transformation from spin states under the action of Lorentz group, since under Lorentz transformations spin states change according to Wigner rotation, which is related to $Z(\Lambda, p)$ as $Z(\Lambda, p) = R^{-1}(\Lambda p)W(\Lambda, p)R(p)$. It is of interest by itself to investigate the implications and ramifications resulting from this difference. However, it is another problem, which is beyond the scope of this paper, and will be expected to be reported elsewhere[38].

Thus a pure one particle state is represented by

$$|\psi\rangle = \sum_{\lambda=\pm\frac{1}{2}} \int d^3\mathbf{p}\psi(\lambda, \mathbf{p})|p;\lambda\rangle \qquad (2)$$

with the normalized condition

$$\sum_{\lambda=\pm\frac{1}{2}} \int d^3\mathbf{p}|\psi(\lambda, \mathbf{p})|^2 = 1. \qquad (3)$$

It is noteworthy that this normalized state with a superposition of various momenta represents a more physical reality since a particle has no definite momentum in general, although for convenience the momentum eigenstates are extensively employed in textbooks on high energy physics and quantum field theory. Then the reduced helicity density matrix associated with the above normalized state is obtained by tracing out the momentum degree of freedom, i.e.,

$$\begin{aligned}
\rho &= Tr_{\mathbf{p}}[|\psi\rangle\langle\psi|] = \int d^3\mathbf{p}\langle\mathbf{p}|\psi\rangle\langle\psi|\mathbf{p}\rangle \\
&= \sum_{\lambda,\tilde{\lambda}} \int d^3\mathbf{p}[\psi(\lambda, \mathbf{p})\psi^*(\tilde{\lambda}, \mathbf{p})|\lambda\rangle\langle\tilde{\lambda}|]. \qquad (4)
\end{aligned}$$

Here, we have used the orthonormal relation for the helicity states. Later a Lorentz transformation Λ changes the one particle state to

$$\begin{aligned}
|\psi'\rangle &= U(\Lambda)|\psi\rangle \\
&= \sum_{\lambda=\pm\frac{1}{2}} \int d^3\mathbf{p}\sqrt{\frac{(\Lambda p)^0}{p^0}}\psi(\lambda, \mathbf{p})D_{\lambda'\lambda}[Z(\Lambda, p)]|\Lambda p;\lambda'\rangle, \qquad (5)
\end{aligned}$$

and the reduced helicity density matrix to

$$\begin{aligned}
\rho' &= \sum_{\lambda'\tilde{\lambda}'} \int d^3\mathbf{p} \\
&\{D_{\lambda'\lambda}[Z(\Lambda, p)]\psi(\lambda, \mathbf{p})\psi^*(\tilde{\lambda}, \mathbf{p})D^\dagger_{\tilde{\lambda}\tilde{\lambda}'}[Z(\Lambda, p)]|\lambda'\rangle\langle\tilde{\lambda}'|\}. \qquad (6)
\end{aligned}$$

By Eq.(1), $D[Z(\Lambda, p)]$ is always an identity matrix if Λ is a purely spacial rotation transformation. Thus the reduced helicity matrix is completely the same for those inertial observers without relative motion but with different identification of spacial direction. This property essentially stems from the fact that the helicity $\frac{(\mathbf{p}\cdot\mathbf{J})}{|\mathbf{p}|}$ remains invariant under a purely spacial rotation transformation[39]. Furthermore, note that any Lorentz transformation can always be decomposed into the product of a pure boost and a pure rotation, we next shall concentrate on what happens to the reduced helicity matrix and the corresponding helicity entanglement entropy when Λ is a pure boost transformation. Especially, taking into account that those pure boost transformations are similarly equivalent with one another by rotations, which means the reduced helicity density matrix only depends on the magnitude of velocity of relative motion between inertial observers, now we shall only need to consider pure boost transformations along the z axis.

In the special case mentioned above, set

$$\Lambda = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}, \eta \leq 0, \qquad (7)$$

and

$$p = m[\cosh\tau, \sinh\tau(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)], \tau \geq 0, \qquad (8)$$

then employing Eq.(1), we obtain

$$\begin{aligned}
D[Z(\Lambda, p)] &= \begin{pmatrix} e^{-\frac{\alpha}{2}} & 0 \\ 0 & e^{\frac{\alpha}{2}} \end{pmatrix} \times \\
&\begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \times \\
&\begin{pmatrix} e^{\frac{\eta}{2}} & 0 \\ 0 & e^{-\frac{\eta}{2}} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix} \times \\
&\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{\frac{\tau}{2}} & 0 \\ 0 & e^{-\frac{\tau}{2}} \end{pmatrix}, \qquad (9)
\end{aligned}$$

where α and β satisfy

$$\cosh\alpha = \cosh\eta\cosh\tau + \sinh\eta\sinh\tau\cos\theta, \alpha \geq 0, \quad (10)$$

and

$$\cos\beta = \frac{\sinh\eta\cosh\tau + \cosh\eta\sinh\tau\cos\theta}{\sqrt{\sinh^2\tau\sin^2\theta + (\sinh\eta\cosh\tau + \cosh\eta\sinh\tau\cos\theta)^2}} \quad (11)$$

with $\pi \geq \beta \geq 0$, respectively.

As an example, consider a particle prepared in the eigenstate with helicity $\frac{1}{2}$, i.e., right handed state, with respect to some original inertial reference frame, which obviously implies that the corresponding helicity entropy is zero, due to $\psi(-\frac{1}{2}, \mathbf{p}) = 0$. If that particle is described in another inertial reference frame moving with velocity $v = -\tanh\eta$ along the z axis of the original one, then substituting Eq.(9) into Eq.(6), and after straightforward but lengthy calculations, the new reduced helicity density matrix can be obtained as

$$\rho' = \int d^3\mathbf{p}\frac{|\psi(\frac{1}{2}, \mathbf{p})|^2}{2}H, \quad (12)$$

where

$$\begin{aligned}
H_{11} &= 1 + \cosh\eta\sin\beta\sin\theta + \cos\beta\cos\theta, \\
H_{12} &= e^{-\alpha}(\sinh\eta\sin\theta + \cosh\eta\cos\beta\sin\theta - \sin\beta\cos\theta), \\
H_{21} &= e^{\alpha}(-\sinh\eta\sin\theta + \cosh\eta\cos\beta\sin\theta - \sin\beta\cos\theta), \\
H_{22} &= 1 - \cosh\eta\sin\beta\sin\theta - \cos\beta\cos\theta. \quad (13)
\end{aligned}$$

For simplicity but without loss of generality, consider in particular the case where the wave function is a Gaussian, i.e.,

$$\psi(\frac{1}{2}, \mathbf{p}) = \pi^{-\frac{3}{4}}\sigma^{-\frac{3}{2}}e^{-\frac{\mathbf{p}^2}{2\sigma^2}}, \quad (14)$$

where σ is the distribution width. However, different from the spin case investigated in [8], the later calculations, especially the integral in Eq.(12), can not be carried out analytically. Thus note that the Von Neumann entropy formula reads

$$S = -tr(\rho\log_2\rho) = -\sum_{i=1,2}\rho_i\log_2\rho_i, \quad (15)$$

where $\{\rho_i\}$ are the eigenvalues of the reduced density matrix ρ, we now resort to numerical methods to perform all calculations. The corresponding results are illustrated in Fig.1.

As shown in Fig.1, with the increase of speed of inertial observers, the variation of the corresponding helicity entanglement entropy demonstrates remarkably different behaviors from that for the spin case investigated in [8]. In particular, for the limiting case of sharp momenta which corresponds to the small width-mass ratio $\frac{\sigma}{m}$, the helicity entanglement entropy blows up from zero and rapidly saturates. Speaking specifically, it arrives

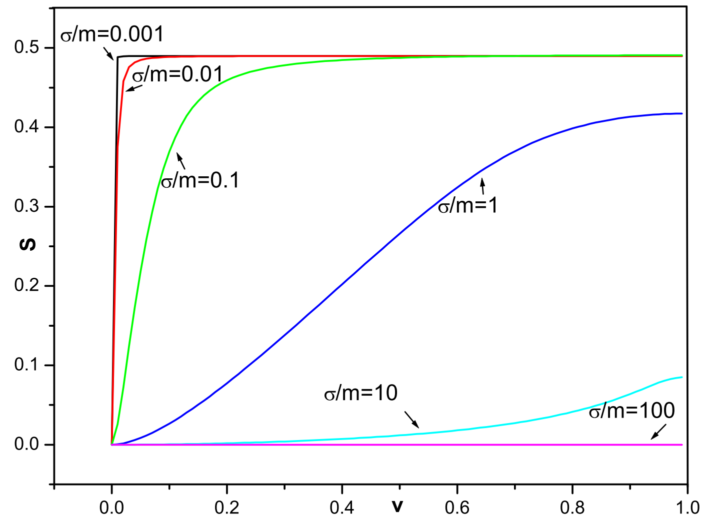

FIG. 1: The helicity entropy S as a function of the speed v of inertial observers with respect to the original inertial reference frame.

at a constant value(about 0.5, half of the entropy value for the maximal entangled Bell states) at small velocities of observers, and then remains nearly invariant regardless of the increase of speed of observers. While the spin entropy always remains vanishing, as mentioned in the beginning[8]. On the other hand, for the large width-mass ratio limiting case, the resultant helicity entanglement entropy remains zero.

Obviously this novel phenomenon differs greatly from that related to the spin case, and unique to the helicity considered here[8, 9, 10, 11, 12]. The key physics underlying this seemingly odd phenomenon lies in the following fact: With the smaller width-mass ratio $\frac{\sigma}{m}$, i.e., sharper momentum distribution, the helicity of the prepared particle becomes more sensitive to Lorentz boost, due to the concentration of its momentum in a smaller neighborhood around zero. In other words, for the smaller width-mass ratio case, the smaller speed of observers is needed to make the flip of helicity from right to left saturate such that the corresponding helicity entropy reaches the saturated value.

In summary, the transformation of the reduced helicity density matrix under Lorentz group is calculated for a massive spin $\frac{1}{2}$ particle. Especially, we have investigated the helicity entropy for a Gaussian one particle state appealing to numerical computation. Our results show that the helicity entanglement entropy is not an invariant scalar, which is the same as the previously considered spin entropy. Nevertheless, as the speed of inertial observers increases, the specific variation of helicity entropy demonstrates a surprisingly distinct behavior from that of spin entropy, which essentially originates from

the fact that the helicity states differ significantly from the spin states in transformation property under Lorentz boost. Put it another way, unlike the spin, the helicity can be more readily flipped by relatively small speed of observers when the momentum distribution is sharp enough.

Associated with this unique feature of helicity, the theoretical implications in quantum information theory and experimental ramifications in high energy physics both need to be further investigated. In addition, our present calculations are restricted into a specific case in a general framework, i.e., a purely Gaussian one particle state. A direct but non-trivial generalization is to adopt other momentum distributions. In addition, more attractive issues involve multi-particle state entanglement, and distillable entanglement of mixed state for the helicity, as developed for the spin case.

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[39] Note that the reduced spin density matrix changes as $\rho' = D(\Lambda)\rho D^{\dagger}(\Lambda)$ for a purely spacial rotation transformation Λ. However, the corresponding spin entropy remains invariant under such a rotation.